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Trajectory attractor and global attractor for a two-dimensional incompressible non-Newtonian fluid ☆

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Abstract

The authors construct the trajectory attractor and global attractor for an autonomous two-dimensional non-Newtonian fluid.

Keywords: Trajectory attractor; Global attractor; Non-Newtonian fluid

1. Introduction and main result

In this paper we construct the trajectory attractor and global attractor for the following autonomous non-Newtonian fluid

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \cdot \tau = g(x), \quad x = (x_1, x_2) \in \Omega,\ t > 0, \tag{1.1}$$

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where Ω is a smooth bounded subset of $\mathbf{R}^2$; the unknown function $u = u(x,t) = (u_1, u_2)$ represents the velocity of the fluid; $g(x) = (g_1, g_2)$ is the external force; and

$$\tau = (\tau_{ij})_{2\times 2}, \quad \text{where } \tau_{ij} = p\delta_{ij} - 2\mu_0\big(\varepsilon + |e|^2\big)^{-\frac{\alpha}{2}} e_{ij} + 2\mu_1 \Delta e_{ij},\ i,j = 1,2, \tag{1.2}$$

stands for the stress tensor and p is the pressure,

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad |e|^2 = \sum_{i,j=1}^{2} |e_{ij}|^2,$$

and μ_0, μ_1, α, ε are parameters which in general depend on the temperature and pressure. In this paper we assume that the four parameters are positive. Equation (1.1) describes the motion of an isothermal, incompressible viscous fluid. In (1.2) the stress tensor τ_{ij} depends non-linearly on e_{ij} and the fluid is called a non-Newtonian one. If the stress tensor τ_{ij} depends linearly on e_{ij} then we say the fluid satisfies the Stokes Law and the corresponding fluid is called a Newtonian one. Obviously, if $\alpha = \mu_1 = 0$, then (1.1) turns into the well-known Navier–Stokes equation. If $\mu_1 = \mu_0 = 0$, then (1.1) reduces to the famous Euler equation. They both are Newtonian fluids.

From the viewpoint of physics, the initial boundary value problem of (1.1) can be formulated as following:

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla p = \nabla\cdot\big(2\mu_0\big(\varepsilon + |e|^2\big)^{-\frac{\alpha}{2}} e - 2\mu_1\Delta e\big) + g(x), \quad x \in \Omega,\ t > 0, \tag{1.3}$$

$$\nabla\cdot u = 0, \quad x\in\Omega,\ t \geqslant 0, \tag{1.4}$$

$$u = 0, \qquad \tau_{ijk} n_j n_k = 0, \quad x \in \partial\Omega,\ t\geqslant 0, \tag{1.5}$$

$$u|_{t=0} = u_0, \tag{1.6}$$

where $\tau_{ijk} = 2\mu_1 \frac{\partial e_{ij}}{\partial x_k}$ (here and below $k = 1,2$) and $n = (n_1, n_2)$ denotes the exterior unit normal to the boundary $\partial\Omega$. The first condition in (1.5) represents the usual no-slip condition associated with a viscous fluid, while the second one expresses the fact that the first moments of the traction vanish on $\partial\Omega$; it is a direct consequence of the principle of virtual work. We refer to [1–7] and references therein for detailed background. There are many works concerning the unique existence, regularity and long time behavior of solutions to the above problem (1.3)–(1.6) or its associated version (see [8–18]).

For convenience, we introduce some spaces. Set

$$\begin{aligned}
&\mathscr{V} = \big\{\varphi = (\varphi_1,\varphi_2) \in C_0^\infty(\overline{\Omega}) \times C_0^\infty(\overline{\Omega}),\ \nabla\cdot\varphi = 0 \text{ in } \Omega,\ \varphi = 0 \text{ on } \partial\Omega\big\},\\
&H = \text{closure of } \mathscr{V} \quad \text{in } L^2(\Omega)\times L^2(\Omega), \qquad H' = \text{dual space of } H,\\
&V = \text{closure of } \mathscr{V} \quad \text{in } H^2(\Omega)\times H^2(\Omega), \qquad V' = \text{dual space of } V.
\end{aligned}$$

$(\cdot,\cdot)$ denotes the inner product in H and $\langle\cdot,\cdot\rangle$ stands for the dual pairing between V and V'; also, we set $H^\eta = (-\Delta)^{-\eta/2}H$ ($\eta \geqslant 0$ and the Laplace operator Δ is taken with zero boundary conditions $u|_{\partial\Omega} = 0$) and use $H^{-\eta}$ to denote the dual space of H^η. In the whole paper, we take $0 < \eta \leqslant 2$ and thus the embedding $H \hookrightarrow H^{-\eta}$ is compact. By excluding the pressure p, we can rewrite the weak version of (1.3)–(1.6) in the solenoidal vector fields as follows (see [6,14]):

$$\frac{\partial u}{\partial t} + 2\mu_1 Au + B(u) + N(u) = g, \quad t > 0, \tag{1.7}$$

$$u(0) = u_0. \tag{1.8}$$

The operators A, $B(\cdot)$ and $N(\cdot)$ appearing in (1.7) will be introduced in Section 2.

Nowadays, much attention has been paid to the study of the dynamical systems (see [19,20, 26,27]). In book [19] Temam studied systematically the notation of global attractor, as well as many concrete autonomous equations arising in mathematical physics. Later on, Chepyzhov and Vishik [20] presented a general method to construct the trajectory attractor and global attractor for the non-autonomous systems, even in the absence of uniqueness of solution. With the notation of global attractor (see [19]), Li and Zhao established the existences of L^2-compact and H^2-compact global attractors for the solution operator semigroup corresponding to (1.7)–(1.8) when $g \in H$ and the parameter $\alpha \in (0, 1)$ (see [15–17]). The aim of this paper is to construct the trajectory attractor and global attractor for system (1.7) when $g \in V'$ and $\alpha > 0$.

To state our main result clearly, we introduce some notations. Firstly, we specify the definition of solutions to problem (1.7)–(1.8). A function $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ is called a weak solution of (1.7)–(1.8) on the interval $[0, T]$ if $u(x, t)$, together with its derivative $\partial_t u(t)$, satisfies (1.7) and (1.8) in the sense of distributions in $\mathcal{D}'(0, T; V')$ (see [25]). We can prove by using Galerkin method that the Cauchy problem (1.7)–(1.8) has at least one solution $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ defined on the interval $[0, T]$ ($\forall T \geqslant 0$) and satisfying the following energy equality

$$\frac{1}{2}\frac{d}{dt}\big(u(t), u(t)\big) + 2\mu_1\langle Au(t), u(t)\rangle + \langle N\big(u(t)\big), u(t)\rangle = \langle g, u(t)\rangle, \quad \forall t \in [0, T], \tag{1.9}$$

in the following sense:

$$\begin{aligned} &-\frac{1}{2}\int_0^T \|u(t)\|^2 \psi'(t)\,dt + 2\mu_1 \int_0^T \langle Au(t), u(t)\rangle\psi(t)\,dt + \int_0^T \langle N\big(u(t)\big), u(t)\rangle\psi(t)\,dt \\ &\quad = \int_0^T \langle g, u(t)\rangle\psi(t)\,dt, \quad \forall \psi \in C_0^\infty\big([0, T]\big),\ \forall T \geqslant 0. \end{aligned} \tag{1.10}$$

In this paper, we use Π_+ to denote the restriction operator (with respect to time variable) to the semi-infinite interval $\mathbf{R}_+$. Analogously, Π_T stands for the restriction operator to the interval $[0, T]$. For example, if $u(\cdot) \in C(\mathbf{R}_+; H^{-\eta}) \cap L^\infty(\mathbf{R}_+; H)$, then $\Pi_T u(\cdot) \in C([0, T]; H^{-\eta}) \cap L^\infty(0, T; H)$; $\Pi_T u(t) = u(t)$ if $t \in [0, T]$.

Definition 1.1. The trajectory space $\mathcal{T}^+$ of Eq. (1.7) consists of functions $u \in L^\infty(\mathbf{R}_+; H) \cap L^2_{\mathrm{loc}}(\mathbf{R}_+; V)$ such that for $\forall T \geqslant 0$ the function $\Pi_T u(t)$ is a weak solution of (1.7) on $[0, T]$ and $\Pi_T u(t)$ satisfies (1.10).

We will study the translation semigroup $\{S(t)\}_{t\geqslant 0}$ acting on the space $\mathcal{T}^+$ in the positive semi-axis according to the formula: $S(t)u(\cdot) = u(t + \cdot)$, $t \geqslant 0$. If a function $u \in C(\mathbf{R}; H^{-\eta}) \cap L^\infty(\mathbf{R}; H)$ and $\Pi_+ u(\cdot + h) \in \mathcal{T}^+$ holds for $\forall h \in \mathbf{R}$, then we say $u(t)$ is a complete trajectory of Eq. (1.7) with $t \in \mathbf{R}$. The union of complete trajectories of Eq. (1.7) with $t \in \mathbf{R}$ is called the kernel of Eq. (1.7) in the space $L^\infty(\mathbf{R}; H)$ and we use $\mathcal{K}$ to denote the kernel in this paper. If $\mathcal{B} \subset \mathcal{T}^+$, then $\mathcal{B}(t) = \{u(t){:}\ u \in \mathcal{B}\} \subseteq H$ is called the section of $\mathcal{B}$ at time $t \geqslant 0$. Similarly, we define $\mathcal{K}(t) = \{u(t){:}\ u \in \mathcal{K}\} \subseteq H$.

Definition 1.2. A set $\mathcal{A}^{\mathrm{tr}} \subseteq \mathcal{T}^+$ is called the trajectory attractor of Eq. (1.7) with respect to the topology $C_{\mathrm{loc}}(\mathbf{R}_+; H^{-\eta})$ if

(i) $\mathcal{A}^{\mathrm{tr}}$ is compact in $C_{\mathrm{loc}}(\mathbf{R}_+; H^{-\eta})$ and bounded in $L^\infty(\mathbf{R}_+; H)$;
(ii) $S(t)\mathcal{A}^{\mathrm{tr}} = \mathcal{A}^{\mathrm{tr}}$, $\forall t \geqslant 0$;
(iii) for any bounded (in $L^\infty(\mathbf{R}_+; H)$ norm) set $\mathcal{B} \subset \mathcal{T}^+$ and $\forall T \geqslant 0$,

$$\lim_{t\to+\infty} \operatorname{dist}_{C([0,T];H^{-\eta})}\big(\Pi_T S(t)\mathcal{B}, \Pi_T \mathcal{A}^{\mathrm{tr}}\big) = 0, \tag{1.11}$$

here and below $\operatorname{dist}_{\mathcal{M}}(X, Y) = \sup_{x\in X} \inf_{y\in Y} \operatorname{dist}_{\mathcal{M}}(x, y)$ denotes the Hausdorff semi-distance from $X \subset \mathcal{M}$ to $Y \subset \mathcal{M}$ in the metric space $\mathcal{M}$.

The main result of this paper reads as follows.

Theorem 1.1. *Let $g \in V'$. Then the non-Newtonian system* (1.7) *possesses a trajectory attractor $\mathcal{A}^{\mathrm{tr}} = \Pi_+\mathcal{K}$ in $\mathcal{T}^+$ satisfying* (i)–(iii) *in Definition* 1.2.

Theorem 1.2. *Suppose $g \in V'$. Then the non-Newtonian systems* (1.7) *possesses a global attractor $\mathcal{A} = \mathcal{A}^{\mathrm{tr}}(0) = \Pi_+\mathcal{K}(0) \subseteq H$ in the following sense*:

(1) *$\mathcal{A}$ is compact in $H^{-\eta}$ and bounded in H*;
(2) *for any bounded* (*in $L^\infty(\mathbf{R}_+; H)$ norm*) *set $\mathcal{B} \subset \mathcal{T}^+$, $\lim_{t\to+\infty} \operatorname{dist}_{H^{-\eta}}(\mathcal{B}(t), \mathcal{A}) = 0$*;
(3) *$\mathcal{A}$ is the minimal set* (*for the inclusion relation*) *among those satisfying* (1) *and* (2).

Remark 1.1. When $0 < \alpha < 1$, [6] showed that the Cauchy problem (1.7)–(1.8) with initial data $u_0 \in H$ is uniquely solvable. But here we only assume that $\alpha > 0$. In this case, it is unknown whether the solution of problem (1.7)–(1.8) is unique. This is caused essentially by the non-linear term $N(\cdot)$ in the equation. However, we can obtain at least one weak solution by Galerkin method when $u_0 \in H$. That is why we study the trajectory attractor and global attractor, instead of the classical global attractor (see [19]).

Throughout this paper we use the standard notation of vector-valued function spaces (see [21]) and their norms. We denote by $H_0^1(\Omega) = W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ and use $\|\cdot\|$ to denote the norm of the space $L^2(\Omega) \times L^2(\Omega)$. If we identify the dual space H' with H itself, then $V \hookrightarrow H = H' \hookrightarrow V'$ with compact embedding. "$\hookrightarrow$" means the embedding between spaces; "$\rightharpoonup$" and "$\rightarrow$" mean weak convergence and strong convergence, respectively. In addition, the summation convention of repeated indices is used in the whole paper.

2. Notations and preliminary results

Firstly, we introduce some operators and put problem (1.3)–(1.6) into the form of (1.7)–(1.8). Set

$$a(u, v) = \sum_{i,j,k=1}^{2} \left(\frac{\partial e_{ij}(u)}{\partial x_k}, \frac{\partial e_{ij}(v)}{\partial x_k}\right) = \sum_{i,j,k=1}^{2} \int_\Omega \frac{\partial e_{ij}(u)}{\partial x_k}\frac{\partial e_{ij}(v)}{\partial x_k}\,dx, \quad u, v \in V. \tag{2.1}$$

Lemma 2.1. [14] *There exist positive constants C_1 and C_2 such that*

$$C_1\|u\|^2_{H^2(\Omega)} \leqslant \sum_{i,j,k=1}^{2} \left(\frac{\partial e_{ij}(u)}{\partial x_k}, \frac{\partial e_{ij}(u)}{\partial x_k}\right) \leqslant C_2\|u\|^2_{H^2(\Omega)}, \quad \forall u \in V. \tag{2.2}$$

By Lemma 2.1 we see that $a(\cdot,\cdot)$ defines a positive symmetric bilinear form on V. Applying the Lax–Milgram lemma we obtain an isometry operator $A \in \mathcal{L}(V, V')$ via

$$\langle Au, v\rangle = a(u, v), \quad \forall u, v \in V.$$

Secondly, we define a continuous trilinear form

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx, \quad u, v, w \in H_0^1(\Omega). \tag{2.3}$$

Since $V \subset H_0^1(\Omega)$, $b(u, v, w)$ is continuous on V. Integrating by parts one can check

$$b(u, v, w) = -b(u, w, v), \quad \forall u, v, w \in V,$$

which implies that $b(u, v, v) = 0$ holds for $\forall u, v \in V$. Now for any $u \in V$,

$$\big\langle B(u), w\big\rangle = b(u, u, w), \quad \forall w \in V \tag{2.4}$$

defines a continuous function $B(u)$ on V. Finally, for $u \in V$, we set $\mu(u) = 2\mu_0(\varepsilon + |e(u)|^2)^{-\frac{\alpha}{2}}$ and define $N(u)$ as

$$\big\langle N(u), v\big\rangle = \sum_{i,j=1}^{2} \int_{\Omega} \mu(u) e_{ij}(u) e_{ij}(v) \, dx, \quad \forall v \in V. \tag{2.5}$$

Lemma 2.2. *If $u \in L^2(0, T; V) \cap L^\infty(0, T; H)$, then Au, $B(u)$, $N(u)$ all belong to $L^2(0, T; V')$.*

Proof. The assertion of this lemma can be found in [15,16]. □

With the above notations, the weak version of (1.3)–(1.6) can be put into a functional equations in $L^2(0, T; V')$ ($\forall T > 0$) in the form of (1.7)–(1.8) (see [6,14]). Moreover, if u belonging to $L^2(0, T; V) \cap L^\infty(0, T; H)$ is a weak solution of (1.7) on the interval $[0, T]$, then from Lemma 2.2 we infer that $\partial_t u(t) \in L^2(0, T; V')$ and thus $u(t) \in C([0, T]; H)$ (see Lemma 2.3 below). Therefore, the initial condition (1.8) with $u_0 \in H$ makes sense.

We next introduce a basic theorem concerning the existence of trajectory attractor for an abstract evolution equation. Let E_0, E be two Banach spaces such that $E \subseteq E_0$. We study the solution $u(t) \in C(\mathbf{R}_+; E_0) \cap L^\infty(\mathbf{R}_+; E)$ of the following evolution equation

$$\frac{\partial u}{\partial t} = F(u), \quad t \geqslant 0. \tag{2.6}$$

We denote the family of solutions of (2.6) by $\mathcal{T}^+$ (*trajectory space*) and assume that $\mathcal{T}^+ \subseteq C(\mathbf{R}_+; E_0) \cap L^\infty(\mathbf{R}_+; E)$. We also assume that $\mathcal{T}^+$ is translation invariant in the sense: for $\forall u(\cdot) \in \mathcal{T}^+$ and $\forall h \geqslant 0$, there holds $u(\cdot + h) \in \mathcal{T}^+$.

Now we introduce some notations that are similar to that introduced in Section 1. If a function $u(t)$ ($t \in \mathbf{R}$) belongs to $C(\mathbf{R}; E_0) \cap L^\infty(\mathbf{R}; E)$ satisfying: $\Pi_+ u(t + h) \in \mathcal{T}^+$ holds for $\forall h \in \mathbf{R}$, then we say $u(t)$ is a complete trajectory of Eq. (2.6) with $t \in \mathbf{R}$. The union of complete trajectories of Eq. (2.6) with $t \in \mathbf{R}$ is called the kernel of Eq. (2.6) in the space $L^\infty(\mathbf{R}; E)$. If $\mathcal{B} \subset \mathcal{T}^+$, then $\mathcal{B}(t) = \{u(t)\colon\ u \in \mathcal{B}\} \subseteq E$ is called the section of $\mathcal{B}$ at time $t \geqslant 0$. Similarly, we define $\mathcal{K}(t) = \{u(t)\colon\ u \in \mathcal{K}\} \subseteq E$.

The following basic theorem was proved in [20,22,23].

Theorem 2.1. *Let the trajectory space $\mathcal{T}^+$ be translation invariant and there exists an attracting set $\varLambda$ such that $\varLambda \subseteq \mathcal{T}^+$, also $\varLambda$ is compact in $C_{\mathrm{loc}}(\mathbf{R}_+; E_0)$ and bounded in $L^\infty(\mathbf{R}_+; E)$. Then* (2.6) *possesses a trajectory attractor $\mathcal{A}^{\mathrm{tr}} \doteq \varPi_+\mathcal{K} \subset \varLambda$ satisfying* (i)–(iii) *in Definition* 1.2 *with $H^{-\eta}$ and H substituted by E_0 and E, respectively. We conclude this section by the following useful result.*

Lemma 2.3. [24] *Let Y be a Banach space and $E \hookrightarrow E_0 \subseteq Y$. Also let the embedding $E \hookrightarrow E_0$ be compact. Set*

$$W_{\infty,p}(0,T;E,Y) = \left\{\phi(t), t \in [0,T]\colon\ \phi(t) \in L^\infty(0,T;E),\ \phi'(t) \in L^p(0,T;Y)\right\},$$

where $p > 1$, with the norm

$$\|\phi\|_{W_{\infty,p}} = \operatorname{ess\,sup}\left\{\left\|\phi(t)\right\|_E\colon\ t \in [0,T]\right\} + \left(\int_0^T \|\phi'\|_Y^p\right)^{1/p}.$$

Then $W_{\infty,p}(0,T;E,Y) \hookrightarrow C([0,T];E_0)$ with compact embedding.

3. Existence of absorbing set

In the sequel, $\mathcal{T}^+$ denotes the trajectory space of Eq. (1.7). In this section we first show some estimates for any trajectory $u \in \mathcal{T}^+$, then we prove the existence of absorbing set and thus obtain the existence of attracting set for the translation semigroup $\{S(t)\}_{t\geqslant 0}$ in $\mathcal{T}^+$. We remark that a set $\varLambda$ satisfying property (iii) of Definition 1.2 is called an attracting set of $\{S(t)\}_{t\geqslant 0}$ in $\mathcal{T}^+$.

By the previous results and definitions, we easily get the following conclusion.

Lemma 3.1.

(i) *For any $u_0 \in H$, there exists a trajectory* (*maybe not unique*) *$u(t) \in \mathcal{T}^+$ such that $u(0) = u_0$;*
(ii) *$\mathcal{T}^+$ is translation invariant under the acting of the translation semigroup $\{S(t)\}_{t\geqslant 0}$, i.e.,*

$$S(t)\mathcal{T}^+ \subseteq \mathcal{T}^+, \quad \forall t \geqslant 0. \tag{3.1}$$

Lemma 3.2. $\mathcal{T}^+ \subseteq C_{\mathrm{loc}}(\mathbf{R}_+; H^{-\eta}) \cap L^\infty(\mathbf{R}_+; H)$.

Proof. On the one hand, for $\forall u(\cdot) \in \mathcal{T}^+$, it is clear that $u(\cdot) \in L^\infty(\mathbf{R}_+; H)$. On the other hand, using Lemma 2.2 and Eq. (1.7) we see that $\partial_t u(\cdot) \in L^2_{\mathrm{loc}}(\mathbf{R}_+; V')$ because $u(\cdot) \in L^\infty(\mathbf{R}_+; H) \cap L^2_{\mathrm{loc}}(\mathbf{R}_+; V)$ and $g \in V'$. Since $H \hookrightarrow H^{-\eta} \subseteq V'$ and the embedding $H \hookrightarrow H^{-\eta}$ is compact, we infer from Lemma 2.3 that $u(\cdot) \in C_{\mathrm{loc}}(\mathbf{R}_+; H^{-\eta})$. The proof of this lemma is completed. □

Lemma 3.3. *For any trajectory $u \in \mathcal{T}^+$, there exist positive constants C, C_0, R_0, and β, which are independent of u, such that*

$$\begin{aligned}
&\left\|S(t)u\right\|_{L^\infty(\mathbf{R}_+;H)} + \left\|S(t)u\right\|_{L^2(0,1;V)} + \left\|S(t)\partial_t u\right\|_{L^2(0,1;V')} \\
&\quad = \operatorname*{ess\,sup}_{s\geqslant t}\left\|u(s)\right\| + \left(\int_t^{t+1}\left\|u(s)\right\|_V^2\,ds\right)^{1/2} + \left(\int_t^{t+1}\left\|\partial_t u(s)\right\|_{V'}^2\,ds\right)^{1/2} \\
&\quad \leqslant C\|u\|_{L^\infty(0,1;H)}e^{-\frac{\beta t}{2}} + C_0\|u\|^2_{L^\infty(0,1;H)}e^{-\beta t} + R_0, \quad \forall t \geqslant 0.
\end{aligned} \tag{3.2}$$

Proof. Using u to take dual pairing $\langle\cdot,\cdot\rangle$ with Eq. (1.7) and integrating with respect to time variable from τ to t ($\tau \leqslant t, \tau, t \in \mathbf{R}_+$), we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\|u(t)\|^2 - \|u(\tau)\|^2\right) \\
&\quad + \sum_{i,j,k=1}^{2} 2\mu_1 \int_\tau^t \left(\frac{\partial e_{ij}(u(s))}{\partial x_k}, \frac{\partial e_{ij}(u(s))}{\partial x_k}\right) ds + \int_\tau^t \langle N(u(s)), u(s)\rangle\, ds \\
&= \int_\tau^t \langle g, u(s)\rangle\, ds \leqslant \int_\tau^t \|g\|_{V'} \|u(s)\|_{H^2(\Omega)}\, ds. \qquad (3.3)
\end{aligned}$$

Since the third term on the left-hand side of (3.3) is non-negative, we have by Lemma 2.1 that

$$\begin{aligned}
&\|u(t)\|^2 + 2C_1\mu_1 \int_\tau^t \|u(s)\|^2_{H^2(\Omega)}\, ds \leqslant \|u(\tau)\|^2 + \frac{2}{C_1\mu_1}\int_\tau^t \|g\|^2_{V'}\, ds, \\
&\tau \leqslant t,\ \tau, t \in \mathbf{R}_+, \qquad (3.4)
\end{aligned}$$

hereafter the constant C_1 comes from Lemma 2.1. Clearly, we have

$$\|u\|^2 \leqslant \|u\|^2_{H^2(\Omega)}, \quad \forall u \in H^2(\Omega). \qquad (3.5)$$

Now from (1.10), Lemma 2.1 and the non-negativity of the term $\int_\tau^t \langle N(u(s)), u(s)\rangle\, ds$, we infer that

$$\begin{aligned}
&-\int_0^{+\infty} \|u(s)\|^2 \psi'(s)\, ds + 2\mu_1 C_1 \int_0^{+\infty} \|u(s)\|^2 \psi(s)\, ds \\
&\quad \leqslant \int_0^{+\infty} \left[\frac{2}{\mu_1 C_1}\|g\|^2_{V'} - 2\mu_1 C_1\left(\|u(s)\|^2_{H^2(\Omega)} - \|u(s)\|^2\right)\right]\psi(s)\, ds \qquad (3.6)
\end{aligned}$$

holds for $\forall \psi(s) \in C_0^\infty(\mathbf{R}_+)$, $\psi(s) \geqslant 0$. At this stage, we need the following lemma which was proved in [20].

Lemma 3.4. *Let $y(s), K(s) \in L^1_{\mathrm{loc}}(0, +\infty)$ and let*

$$-\int_0^{+\infty} y(s)\phi'(s)\, ds + \beta \int_0^{+\infty} y(s)\phi(s)\, ds \leqslant \int_0^{+\infty} K(s)\phi(s)\, ds$$

holds for any $\phi(s) \in C_0^\infty(\mathbf{R}_+)$, $\phi(s) \geqslant 0$, where $\beta \in \mathbf{R}$. Then for any $t, \tau \in \mathbf{R}_+$, $t \geqslant \tau$ there holds

$$y(t)e^{\beta t} - y(\tau)^{\beta\tau} \leqslant \int_\tau^t K(s)e^{\beta s}\, ds.$$

We continue to prove Lemma 3.3. Applying Lemma 3.4 for

$$\beta = 2\mu_1 C_1, \qquad y(s) = \|u(s)\|^2,$$
$$K(s) = \frac{2}{\mu_1 C_1}\|g\|_{V'}^2 - 2\mu_1 C_1\big(\|u(s)\|_{H^2(\Omega)}^2 - \|u(s)\|^2\big),$$

we get

$$\|u(t)\|^2 e^{\beta t} - \|u(\tau)\|^2 e^{\beta\tau} + 2\mu_1 C_1 \int_\tau^t \big(\|u(s)\|_{H^2(\Omega)}^2 - \|u(s)\|^2\big)e^{\beta s}\,ds$$
$$\leqslant \frac{2}{\mu_1 C_1}\int_\tau^t \|g\|_{V'}^2 e^{\beta s}\,ds = \frac{2\|g\|_{V'}^2}{\mu_1 C_1 \beta}\big(e^{\beta t} - e^{\beta\tau}\big), \quad t \geqslant \tau,\ t,\tau \in \mathbf{R}_+. \tag{3.7}$$

Using (3.5) and (3.7) we easily get

$$\|S(t)u\|_{L^\infty(\mathbf{R}_+;H)} \leqslant \frac{\sqrt{2}\|g\|_{V'}}{\sqrt{\mu_1 C_1 \beta}} + \|u\|_{L^\infty(0,1;H)} e^{-\frac{\beta t}{2}} \doteq \|u\|_{L^\infty(0,1;H)} e^{-\frac{\beta t}{2}} + R_1, \tag{3.8}$$

where $R_1 = \frac{\sqrt{2}\|g\|_{V'}}{\sqrt{\mu_1 C_1 \beta}}$ is independent of u. Then it follows from (3.4) and (3.8) that

$$\|S(t)u\|_{L^2(0,1;V)} = \left(\int_t^{t+1} \|u(s)\|_{H^2(\Omega)}^2\,ds\right)^{1/2} \leqslant \left(\frac{\|u(t)\|^2}{2\mu_1 C_1} + \frac{\|g\|_{V'}^2}{(\mu_1 C_1)^2}\right)^{1/2}$$
$$\leqslant \frac{\|u(t)\|}{\sqrt{2\mu_1 C_1}} + \frac{\|g\|_{V'}}{\mu_1 C_1} \leqslant \frac{R_1 + \|u\|_{L^\infty(0,1;H)} e^{-\frac{\beta t}{2}}}{\sqrt{2\mu_1 C_1}} + \frac{\|g\|_{V'}}{\mu_1 C_1}$$
$$= C_3\|u\|_{L^\infty(0,1;H)} e^{-\frac{\beta t}{2}} + R_2, \tag{3.9}$$

where $C_3 = \frac{1}{\sqrt{2\mu_1 C_1}}$ and $R_2 = \frac{R_1}{\sqrt{2\mu_1 C_1}} + \frac{\|g\|_{V'}}{\mu_1 C_1}$ are both independent of u. According to the definition of $B(u)$, we use (3.8) and (3.9), as well as Hölder inequality, to obtain

$$\left(\int_t^{t+1} \|B(u(s))\|_{V'}^2\,ds\right)^{1/2} \leqslant \left(\int_t^{t+1} \|u(s)\|^2 \|\nabla u(s)\|^2\,ds\right)^{1/2}$$
$$\leqslant \sup_{s\in[t,t+1]} \|u(s)\| \cdot \left(\int_t^{t+1} \|u(s)\|_{H^2(\Omega)}^2\,ds\right)^{1/2}$$
$$\leqslant \big(\|u\|_{L^\infty(0,1;H)} e^{-\frac{\beta t}{2}} + R_1\big)\big(C_3\|u\|_{L^\infty(0,1;H)} e^{-\frac{\beta t}{2}} + R_2\big)$$
$$= C_3\|u\|_{L^\infty(0,1;H)}^2 e^{-\beta t} + C_4\|u\|_{L^\infty(0,1;H)} e^{-\frac{\beta t}{2}} + R_3, \tag{3.10}$$

where $C_4 = C_3 R_1 + R_2$ and $R_3 = R_1 R_2$ are both independent of u. Similarly, we have

$$\left(\int_t^{t+1} \|N(u(s))\|_{V'}^2\,ds\right)^{1/2}$$
$$\leqslant \mu_0 \varepsilon^{-\frac{\alpha}{2}} \left(\int_t^{t+1} \|\nabla u(s)\|^2\,ds\right)^{1/2} \leqslant \mu_0 \varepsilon^{-\frac{\alpha}{2}} \left(\int_t^{t+1} \|u(s)\|_V^2\,ds\right)^{1/2}$$
$$\leqslant \mu_0 \varepsilon^{-\frac{\alpha}{2}}\big(C_3\|u\|_{L^\infty(0,1;H)} e^{-\frac{\beta t}{2}} + R_2\big) = C_5\|u\|_{L^\infty(0,1;H)} e^{-\frac{\beta t}{2}} + R_4, \tag{3.11}$$

where $C_5 = C_3\mu_0\varepsilon^{-\frac{\alpha}{2}}$ and $R_4 = \mu_0\varepsilon^{-\frac{\alpha}{2}}R_2$ both are independent of u. Since A is an isometry operator from V to V', Eq. (1.7) implies that

$$\begin{aligned}\left(\int_t^{t+1}\left\|\partial_t u(s)\right\|_{V'}^2\,ds\right)^{1/2} &\leqslant 2\mu_1\left(\int_t^{t+1}\left\|Au(s)\right\|_{V'}^2\,ds\right)^{1/2} + \left(\int_t^{t+1}\left\|B\big(u(s)\big)\right\|_{V'}^2\,ds\right)^{1/2} \\ &\quad + \left(\int_t^{t+1}\left\|N\big(u(s)\big)\right\|_{V'}^2\,ds\right)^{1/2} + \left(\int_t^{t+1}\|g\|_{V'}^2\,ds\right)^{1/2} \\ &\leqslant 2\mu_1\left(\int_t^{t+1}\left\|u(s)\right\|_V^2\,ds\right)^{1/2} + C_3\|u\|_{L^\infty(0,1;H)}^2e^{-\beta t} + R_3 + R_4 \\ &\quad + C_4\|u\|_{L^\infty(0,1;H)}e^{-\frac{\beta t}{2}} + C_5\|u\|_{L^\infty(0,1;H)}e^{-\frac{\beta t}{2}} + \|g\|_{V'} \\ &\leqslant C_3\|u\|_{L^\infty(0,1;H)}^2e^{-\beta t} + C_6\|u\|_{L^\infty(0,1;H)}e^{-\frac{\beta t}{2}} + R_5, \end{aligned} \tag{3.12}$$

where $C_6 = 2\mu_1C_3 + C_4 + C_5$ and $R_5 = 2\mu_1R_2 + R_3 + R_4 + \|g\|_{V'}$ are both independent of u. Using (3.8), (3.9) and (3.12), we obtain (3.2). The proof is completed. □

We next use the estimate obtained in Lemma 3.3 to construct the attracting set for $\{S(t)\}_{t\geqslant 0}$ in $\mathcal{T}^+$.

Lemma 3.5. *There exists a bounded* (*in the norm of* $L^\infty(\mathbf{R}_+; H)$) *absorbing set* $\Lambda \subset \mathcal{T}^+$*, i.e., for any bounded* (*in the norm of* $L^\infty(\mathbf{R}_+; H)$) *subset* $\mathcal{B} \subset \mathcal{T}^+$*, there exists a time* $t_0 = t_0(\mathcal{B})$ *such that* $S(t)u \in \Lambda$*,* $\forall u \in \mathcal{B}$*,* $\forall t \geqslant t_0$.

Proof. Set

$$\Lambda = \Big\{u \in \mathcal{T}^+\colon \sup_{t\geqslant 0}\big\{\|u\|_{L^\infty(t,t+1;H)} + \|\partial_t u\|_{L^2(t,t+1;V')}\big\} \leqslant 3R_0\Big\}, \tag{3.13}$$

where R_0 is the positive constant comes from Lemma 3.3. We shall prove that Λ is a bounded absorbing set (thus is a bounded attracting set) for the translation semigroup $\{S(t)\}_{t\geqslant 0}$. Indeed, let $\mathcal{B}$ be a bounded (in $L^\infty(\mathbf{R}_+; H)$ norm) subset of $\mathcal{T}^+$. Then from (3.2) we infer that for $\forall u \in \mathcal{B} \subset \mathcal{T}^+$, there exists a $t_0 = t_0(\mathcal{B}) \geqslant 0$ such that $C\|u\|_{L^\infty(0,1;H)}e^{-\frac{\beta t}{2}} \leqslant R_0$ and $C_0\|u\|_{L^\infty(0,1;H)}^2e^{-\beta t} \leqslant R_0$ as long as $t \geqslant t_0$. Therefore,

$$\|u\|_{L^\infty(t,t+1;H)} + \|\partial_t u\|_{L^2(t,t+1;V')} \leqslant 3R_0, \quad \forall t \geqslant t_0, \tag{3.14}$$

and we get $S(t)\mathcal{B} \subseteq \Lambda$, $\forall t \geqslant t_0$, which implies that Λ is an absorbing set for $\{S(t)\}_{t\geqslant 0}$ in $\mathcal{T}^+$. Obviously, Λ is bounded (in $L^\infty(\mathbf{R}_+; H)$ norm) in $\mathcal{T}^+$. The proof is completed. □

4. Trajectory attractor and global attractor

Before proving our main result, we establish the following lemma.

Lemma 4.1. *Let* $\{u_n\}$ *be a bounded* (*in the norm of* $L^\infty(\mathbf{R}_+; H)$) *sequence in* $\mathcal{T}^+$ *and there exists a function* $u^* \in C_{\mathrm{loc}}(\mathbf{R}_+; H^{-\eta})$ *such that*

$$u_n \to u^* \quad (\textit{as } n \to \infty) \textit{ strongly in } C_{\mathrm{loc}}\big(\mathbf{R}_+; H^{-\eta}\big). \tag{4.1}$$

Then $u^* \in \mathcal{T}^+$.

Proof. We need to show that $u^* \in L^\infty(\mathbf{R}_+; H) \cap L^2_{\mathrm{loc}}(\mathbf{R}_+; V)$, and for $\forall T > 0$, $\Pi_T u^*(t)$ is a weak solution of (1.7) on the interval $[0, T]$ satisfying the energy equality (1.10). Indeed, since $\{u_n\} \subset \mathcal{T}^+$ and is bounded in $L^\infty(\mathbf{R}_+; H)$, by (3.2) we conclude that $\{u_n\}$ is bounded in $L^2_{\mathrm{loc}}(\mathbf{R}_+; V)$ and $\{\partial_t u_n\}$ is bounded in $L^2_{\mathrm{loc}}(\mathbf{R}_+; V')$. Using the diagonal procedure we deduce that there exists a function $u \in L^\infty(\mathbf{R}_+; H) \cap L^2_{\mathrm{loc}}(\mathbf{R}_+; V)$ and a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that

$$\Pi_T u_{n'} \rightharpoonup \Pi_T u \quad \text{weakly in } L^2(0, T; V) \text{ as } n' \to \infty; \tag{4.2}$$

$$u_{n'} \rightharpoonup u \quad \text{weakly star in } L^\infty(\mathbf{R}_+; H) \text{ as } n' \to \infty; \tag{4.3}$$

$$\partial_t \Pi_T u_{n'} \rightharpoonup \partial_t \Pi_T u \quad \text{weakly in } L^2(0, T; V') \text{ as } n' \to \infty. \tag{4.4}$$

Obviously, $\partial_t u \in L^2_{\mathrm{loc}}(\mathbf{R}_+; V')$. By Lemma 2.3 we obtain $\Pi_T u \in C([0, T]; H^{-\eta})$ since the embedding $H \hookrightarrow H^{-\eta}$ is compact. From (4.1) and the uniqueness of limit we have $u = u^*$. Next we verify that $\Pi_T u^*$ is a weak solution of (1.7) on the interval $[0, T]$ satisfying (1.10). To this end, we prove the following relations:

$$A\Pi_T u_{n'} \rightharpoonup A\Pi_T u^* \quad \text{weakly in } L^2(0, T; V') \text{ as } n' \to \infty; \tag{4.5}$$

$$B(\Pi_T u_{n'}) \rightharpoonup B(\Pi_T u^*) \quad \text{weakly in } L^2(0, T; V') \text{ as } n' \to \infty; \tag{4.6}$$

$$N(\Pi_T u_{n'}) \rightharpoonup N(\Pi_T u^*) \quad \text{weakly in } L^2(0, T; V') \text{ as } n' \to \infty. \tag{4.7}$$

In fact, for $\forall \phi \in L^2(0, T; V) \cap C([0, T]; V)$, we obtain by using (4.2) and by the definition of the operator A that

$$\lim_{n' \to \infty} \int_0^T \langle A\Pi_T u_{n'} - A\Pi_T u^*, \phi\rangle\, dt = \int_0^T \langle \Pi_T u_{n'} - \Pi_T u^*, A\phi\rangle\, dt = 0, \tag{4.8}$$

where we use the fact $A\phi \in L^2(0, T; V')$. At the same time, using (4.2) and the compact embedding $V \hookrightarrow H^1_0(\Omega)$ we have

$$\Pi_T u_{n'} \to \Pi_T u^* \quad \text{strongly in } L^2\big(0, T; H^1_0(\Omega)\big) \text{ as } n' \to \infty. \tag{4.9}$$

Applying Hölder inequality, Gagliardo–Nirenberg inequality and (4.9) we get

$$\begin{aligned}
&\left| \lim_{n' \to \infty} \int_0^T \big\langle B(\Pi_T u_{n'}) - B(\Pi_T u^*), \phi \big\rangle\, dt \right| \\
&\quad \leqslant \lim_{n' \to \infty} \int_0^T \big| b\big(\Pi_T(u_{n'} - u^*), \Pi_T u_{n'}, \phi\big) - b\big(\Pi_T u^*, \Pi_T(u_{n'} - u^*), \phi\big) \big|\, dt \\
&\quad = \lim_{n' \to \infty} \int_0^T \big| -b\big(\Pi_T(u_{n'} - u^*), \phi, \Pi_T u_{n'}\big) + b\big(\Pi_T u^*, \phi, \Pi_T(u_{n'} - u^*)\big) \big|\, dt \\
&\quad \leqslant c \lim_{n' \to \infty} \int_0^T \big\| \Pi_T(u_{n'} - u^*) \big\|^{1/2} \big\| \Pi_T(u_{n'} - u^*) \big\|^{1/2}_{H^1_0(\Omega)} \|\phi\|_{H^1_0(\Omega)} \\
&\qquad \times \|\Pi_T u_{n'}\|^{1/2} \|\Pi_T u_{n'}\|^{1/2}_{H^1_0(\Omega)}\, dt
\end{aligned}$$

$$
\begin{aligned}
&+c\lim_{n'\to\infty}\int_0^T \big\|\Pi_T(u_{n'}-u^*)\big\|^{1/2}\big\|\Pi_T(u_{n'}-u^*)\big\|^{1/2}_{H_0^1(\Omega)}\|\phi\|_{H_0^1(\Omega)} \\
&\quad\times\|\Pi_T u^*\|^{1/2}\|\Pi_T u^*\|^{1/2}_{H_0^1(\Omega)}\,dt \\
&\leqslant c\lim_{n'\to\infty}\big\|\Pi_T(u_{n'}-u^*)\big\|_{L^2(0,T;H_0^1(\Omega))} \\
&\quad\times\big(\|\Pi_T u_{n'}\|_{L^2(0,T;H_0^1(\Omega))}+\|\Pi_T u^*\|_{L^2(0,T;H_0^1(\Omega))}\big)=0, \qquad (4.10)
\end{aligned}
$$

here c is a constant which is independent of $u_{n'}$ and u^*. Similarly we have

$$
\begin{aligned}
&\left|\lim_{n'\to\infty}\int_0^T\big\langle N(\Pi_T u_{n'})-N(\Pi_T u^*),\phi\big\rangle\,dt\right| \\
&\leqslant \lim_{n'\to\infty}2\mu_0\varepsilon^{-\frac{\alpha}{2}}\int_0^T\big\|\Pi_T(u_{n'}-u^*)\big\|_{H_0^1(\Omega)}\|\phi\|_{H_0^1(\Omega)}\,dt \\
&\leqslant 2\mu_0\varepsilon^{-\frac{\alpha}{2}}\lim_{n'\to\infty}\big\|\Pi_T(u_{n'}-u^*)\big\|_{L^2(0,T;H_0^1(\Omega))}\|\phi\|_{L^2(0,T;V)}=0. \qquad (4.11)
\end{aligned}
$$

Since $C([0,T];V)$ is dense in $L^2(0,T;V)$, (4.8), (4.10) and (4.11) imply that (4.5)–(4.7) hold true. By (4.4)–(4.7) we can pass the limit in Eq. (1.7) as $n'\to\infty$. This shows that $\Pi_T u^*$ is a weak solution of Eq. (1.7) on $[0,T]$. The fact that $\Pi_T u^*$ satisfies (1.10) can easily be established by using $\Pi_T u^*$ to take dual pairing $\langle\cdot,\cdot\rangle$ with (1.7). We complete the proof of Lemma 4.1. □

We begin to prove the main result.

Proof of Theorem 1.1. According to Theorem 2.1, Lemmas 3.1 and 3.2, we only need to prove that the absorbing set $\Lambda\subseteq\mathcal{T}^+$ constructed in Lemma 3.5 is bounded in $L^\infty(\mathbf{R}_+;H)$ and compact in $C_{\mathrm{loc}}(\mathbf{R}_+;H^{-\eta})$. Next we only establish that Λ is compact in $C_{\mathrm{loc}}(\mathbf{R}_+;H^{-\eta})$ because the fact that Λ is bounded in $L^\infty(\mathbf{R}_+;H)$ is obvious. Actually, from (3.13) one can see that $\Pi_T\Lambda$ is bounded in $W_{\infty,2}(0,T;H,V')$ and thus $\Pi_T\Lambda$ is pre-compact in $C([0,T];H^{-\eta})$ (thanks to Lemma 2.3). Hence, it suffices to show that $\Pi_T\Lambda$ is closed in $C([0,T];H^{-\eta})$ for any $T>0$. Assume that $\{u_n\}\subset\Lambda$ and $\Pi_T u_n\to\Pi_T u$ strongly in the norm of $C([0,T];H^{-\eta})$ as $n\to\infty$. Since $\{u_n\}$ is bounded in $L^\infty(\mathbf{R}_+;H)$, applying Lemma 4.1 we see $u\in\mathcal{T}^+$. Moreover, in the proof of Lemma 4.1 we know $\partial_t\Pi_T u_n\rightharpoonup\partial_t\Pi_T u$ weakly in $L^2(0,T;V')$ and $u_n\rightharpoonup u$ weakly star in $L^\infty(\mathbf{R}_+;H)$ as $n\to\infty$. Thus we obtain

$$
\begin{aligned}
&\|u\|_{L^\infty(t,t+1;H)}+\|\partial_t u\|_{L^2(t,t+1;V')} \\
&\quad\leqslant\liminf_{n\to\infty}\|u_n\|_{L^\infty(t,t+1;H)}+\liminf_{n\to\infty}\|\partial_t u_n\|_{L^2(t,t+1;V')}\leqslant 3R_0,\quad \forall t\geqslant 0.
\end{aligned}
$$

Therefore, $u\in\Lambda$ and $\Pi_T u\in C([0,T];H^{-\eta})$. The proof is completed. □

Proof of Theorem 1.2. By Theorem 1.1 we see that the non-Newtonian system (1.7) possesses a trajectory attractor $\mathcal{A}^{\mathrm{tr}}=\Pi_+\mathcal{K}$ in $\mathcal{T}^+$, which satisfies (i)–(iii) in Definition 1.2. Using the definition of $\mathcal{A}^{\mathrm{tr}}(t)$ and the strictly invariance of $\mathcal{A}^{\mathrm{tr}}$, we infer that $\mathcal{A}^{\mathrm{tr}}(t)$ is independent of t, i.e., $\mathcal{A}^{\mathrm{tr}}(t_1)=\mathcal{A}^{\mathrm{tr}}(t_2)$ holds for $t_1\neq t_2, t_1,t_2\geqslant 0$. At the same time, since $\mathcal{A}^{\mathrm{tr}}$ is compact in $C_{\mathrm{loc}}(\mathbf{R}_+;H^{-\eta})$ and bounded in $L^\infty(\mathbf{R}_+;H)$, also $\mathcal{A}^{\mathrm{tr}}(0)\subseteq\mathcal{A}^{\mathrm{tr}}$, we see $\mathcal{A}\doteq\mathcal{A}^{\mathrm{tr}}(0)$ is compact in $H^{-\eta}$ and bounded in H. Thus property (1) in Theorem 1.2 holds. Now for any bounded (in $L^\infty(\mathbf{R}_+; H)$ norm) set $\mathcal{B} \subset \mathcal{T}^+$, applying (1.11) we get

$$\lim_{t\to+\infty} \operatorname{dist}_{C([0,T];H^{-\eta})}\big(\Pi_T S(t)\mathcal{B}, \Pi_T \mathcal{A}^{\mathrm{tr}}\big) = 0, \quad \forall T \geqslant 0. \tag{4.12}$$

Noting $\Pi_0\mathcal{A}^{\mathrm{tr}} = \mathcal{A}^{\mathrm{tr}}(0) = \mathcal{A}$ and $\Pi_0 S(t)\mathcal{B} = \mathcal{B}(t)$, we take $T = 0$ in (4.12) and then property (2) in Theorem 1.2 is established. Lastly, we prove property (3). Suppose $\mathcal{A}_1$ is compact in $H^{-\eta}$ and bounded in H, such that $\lim_{t\to+\infty} \operatorname{dist}_{H^{-\eta}}(\mathcal{B}(t), \mathcal{A}_1) = 0$ holds for any bounded (in $L^\infty(\mathbf{R}_+; H)$ norm) set $\mathcal{B} \subset \mathcal{T}^+$. Taking $\mathcal{B} = \mathcal{A}^{\mathrm{tr}}$, we have

$$\lim_{t\to+\infty} \operatorname{dist}_{H^{-\eta}}\big(\mathcal{A}^{\mathrm{tr}}(t), \mathcal{A}_1\big) = \lim_{t\to+\infty} \operatorname{dist}_{H^{-\eta}}\big(\mathcal{A}^{\mathrm{tr}}(0), \mathcal{A}_1\big) = \operatorname{dist}_{H^{-\eta}}(\mathcal{A}, \mathcal{A}_1) = 0.$$

Since $\mathcal{A}_1$ is compact in $H^{-\eta}$, $\mathcal{A}_1$ is closed in $H^{-\eta}$ and we obtain $\mathcal{A} \subseteq \bar{\mathcal{A}}_1 = \mathcal{A}_1$. The proof of Theorem 1.2 is completed. □

Remark 4.1. The global attractor obtained in Theorem 1.2 is strictly invariant under the acting of the translation semigroup $\{S(t)\}_{t\geqslant 0}$. Indeed, since $\mathcal{A}^{\mathrm{tr}}(t)$ is independent of t, we have

$$S(t)\mathcal{A} = S(t)\mathcal{A}^{\mathrm{tr}}(0) = \mathcal{A}^{\mathrm{tr}}(t) = \mathcal{A}^{\mathrm{tr}}(0) = \mathcal{A}, \quad \forall t \geqslant 0.$$

Remark 4.2. We can use the method here to obtain the similar conclusion for this autonomous three-dimensional non-Newtonian fluid. Also we can construct the uniform trajectory attractor and global attractor for this non-autonomous non-Newtonian fluid.

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